Series 2: solutions

Exercise 1

Use image theory and the fact that the tangentiel E fields in zero on a PEC to:

1. Find the E field radiated by a Hertzian dipole placed at a height h above a ground plane, supporting a current I and oriented orthogonally to the ground ***Adivice***: place the ground plane at $z = 0$ and the dipole at point P *(0, 0, h)* and work in spherical cooridnates *(r, θ)*.
2. Sketch the E field as a function of θ.

Solution:

We first formulate the equivalent prbolem (fig. 1):

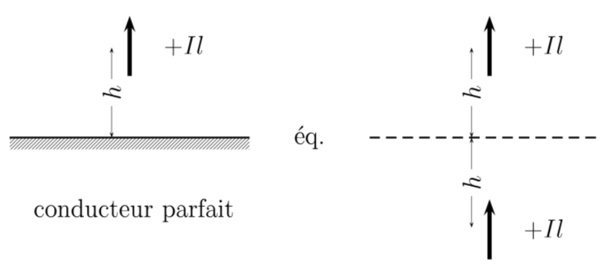


Figure 1. Image theory applied to the vertical Hertzian dipole

We then write the vector integral with respect to the coordinate system depicted in Fig. 2Puis on écrit l'intégrale de rayonnement par rapport au système de coordonnées de la Figure 3.

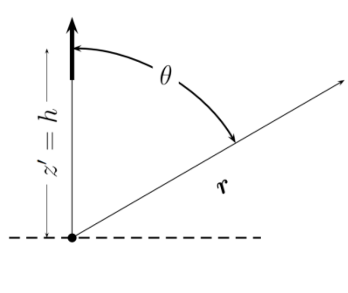


Figure 2. Associated coordinate system

$$\boldsymbol{f}(\theta,\phi)=\int_v \boldsymbol{J}(\boldsymbol{r}')e^{jk\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'}dv'=\hat{\boldsymbol{z}}\left(\int_{h-l/2}^{h+l/2} I\,e^{+jkz'\cos\theta}dz'+\int_{-h-l/2}^{-h+l/2} I\,e^{-jkz'\cos\theta}dz'\right)$$

We obtain $\hat{\boldsymbol{r}}\cdot\boldsymbol{r}'$using $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}}=\cos\theta$ and $\boldsymbol{r}'=h\hat{\boldsymbol{z}}$ (for the original dipole) or $\boldsymbol{r}'=-h\hat{\boldsymbol{z}}$ (for the image). As these dipoles have an infinitesimal length, teh exponetial can be taken out of the integral, yielding:

$$\boldsymbol{f}(\theta,\phi)=\hat{\boldsymbol{z}}\left(Ie^{+jkh\cos\theta}\int_{h-l/2}^{h+l/2}dz'+Ie^{-jkh\cos\theta}\int_{-h-l/2}^{-h+l/2}dz'\right)=\hat{\boldsymbol{z}}2Il\cos(kh\cos\theta)=f_z\hat{\boldsymbol{z}}$$

We find now the component transverse to the propagation (which is in $\boldsymbol{r}$) $\boldsymbol{f}_{trans}=f_\theta\hat{\boldsymbol{\theta}}+f_\phi\hat{\boldsymbol{\phi}}$ with $f_\theta=-\sin\theta\,f_z$ and $f_\phi=0$ and we obtain the electric field :

$$\boldsymbol{E}=-j\frac{Z_c}{2\lambda}\frac{e^{-jkr}}{r}\boldsymbol{f}_{trans}=jIl\frac{Z_c}{\lambda}\sin\theta\cos(kh\cos\theta)\frac{e^{-jkr}}{r}\hat{\boldsymbol{\theta}}$$

The field pattern is independent of φ, as could be expected from the azimuthal symmetry of the problem. We can distinguish two facors in teh expression of the field: the first, $\sin\theta$, is coming from the pattern of the elementary dipole, while the second, $\cos(kh\cos\theta)$, results from the relative position between the dipole and its image.

As we can see on figs 3-5, the pattern has a single zero for $h\le\lambda/4$ (corresponding to the zero in the dipole pattern), as for $\cos(kh\cos\theta)$ equal to zero, its argument needs to be above $\pi/2$ for $\theta\in[0,\pi]$.

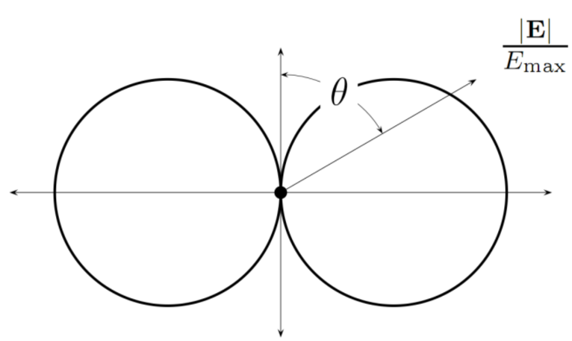


Figure 3. Field pattern of the Hertzian dipole above a ground plane ($h \ll \lambda$).

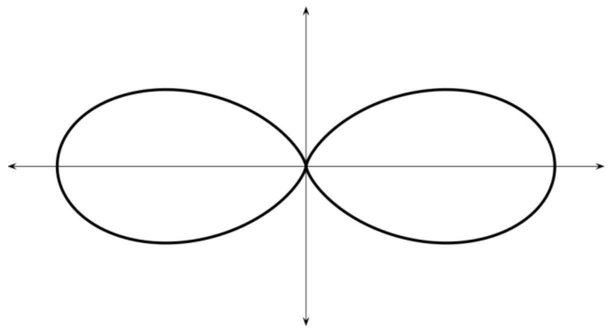


Figure 4. Field pattern of the Hertzian dipole above a ground plane ($h = \lambda/4$).

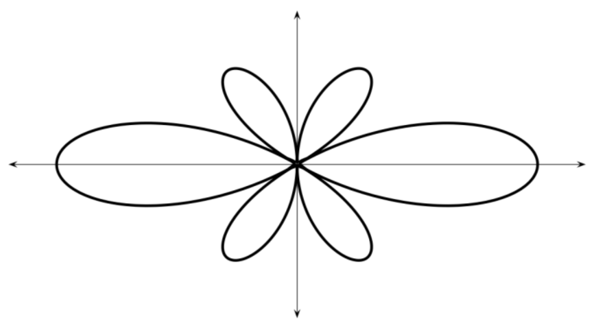


Figure 5 Field pattern of the Hertzian dipole above a ground plane ($h = \lambda/2$).

Exercise 2

The electric field transmitted by an antenna is given by:

$\vec{\mathbf{E}} = \hat{\mathbf{e}}_{\boldsymbol{\theta}} C \cos^n \theta \sin\theta \exp(-jkr)/r$

where C is a constant, n is an integer and the expression is valid in the far field.

Dans cette région, le champ magnétique peut toujours être calculé comme:

$\boldsymbol{H} = (1/Z_0)\hat{\boldsymbol{e}}_{\boldsymbol{r}} \times \boldsymbol{E}$ où Z_0 is a constant, the characteristic impedance of the medium

1. Find the expression of the raidated power density, given by teh amplitude of the Poynting vector $p(r,\theta,\varphi) = \left|\vec{\mathbf{S}}\right| = \left|\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*\right|$
2. Find the total radiated power $P_{rad}(r = R)$, integrating the power density $p(r = R,\theta,\varphi)$ on the spherical surface of radius R Does this power depend on R ?

NOTE: if you find the math difficult, try at last the case $n = 0$

3. Directivity. Compute the directivity, by first computing the mean of the radiated power

density, $p_{iso}(r=R)$ existing on a surface of radius R, as the ratio between the total radiated power and the surface of the sphere. The directivity is then given by $D(\theta,\varphi)=p(r=R,\theta,\varphi)/p_{iso}(r=R)$. Find its maximum value, $D_{\max}$ as a function of n. Give the results in dB

Solution:

1)

In the far field, we have,

$$\boldsymbol{H}=(1/Z_0)\hat{\boldsymbol{e}}_{\boldsymbol{r}}\times\boldsymbol{E}$$

Which in our case yields:

$$\vec{\mathbf{H}}=\frac{1}{Z_c}\hat{\mathbf{e}}_\varphi C\cos^n\theta\sin\theta\frac{e^{-jkr}}{r}$$

The radiated power density [watt/m^2] is then given by $p(r,\theta,\varphi)=\left|\vec{\mathbf{S}}\right|=\left|\vec{\mathbf{E}}\times\vec{\mathbf{H}}^*\right|=\frac{1}{Z_c}\left|\vec{\mathbf{E}}\right|^2=\frac{1}{Z_c}C^2\cos^{2n}\theta\sin^2\theta\frac{1}{r^2}$

1)

The total radiated power is obtained integrating the power densitiy on the spurface of a sphere of radius R:

$$P_{rad}=\iint_S p\,dS$$

In spheircal coordinates, we have $dS=R^2\sin\theta d\theta d\varphi$.

Thus:

$$P_{rad}(r=R)=\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}p(r=R,\theta,\varphi)\,R^2\sin\theta d\theta d\varphi=\frac{C^2}{Z_c}2\pi\int_{\theta=0}^{\pi}\cos^{2n}\theta\sin^2\theta\sin\theta d\theta$$

The integral can be solved using the following substitution: $\sin^2\theta=1-\cos^2\theta$.

We get:

$$P_{rad}(r=R)=\frac{2\pi C^2}{Z_c}\left(\int_0^\pi \cos^{2n}\theta\sin\theta d\theta-\int_0^\pi \cos^{2n+2}\theta\sin\theta d\theta\right)$$

$$=\frac{2\pi C^2}{Z_c}\left(\left.\frac{\cos^{2n+1}\theta}{2n+1}\right|_\pi^0-\left.\frac{\cos^{2n+3}\theta}{2n+3}\right|_\pi^0\right)=\frac{2\pi C^2}{Z_c}\left(\frac{2}{2n+1}-\frac{2}{2n+3}\right)$$

We see that as expected, the total radiated power does not depend at which distance *R* we perform the integration of *p*.

3) Directivity

The mean or isotropic radiated power density (ou isotrope) $p_{iso}(r=R)$ on a spherical surface of radius *R* is given by:

$$p_{iso}(r=R)=\frac{P_{rad}}{4\pi R^2}=\frac{C^2}{Z_c R^2}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$$

And the directivity is defined as :

$$D(\theta,\varphi)=\frac{p(r=R,\theta,\varphi)}{p_{iso}(r=R)}=\frac{\cos^{2n}\theta\sin^2\theta}{\dfrac{1}{2n+1}-\dfrac{1}{2n+3}}=D(\theta)$$

Maximal directivty

- for $n=0$

  $D(\theta)=\frac{3}{2}\sin^2\theta$ and $D_{\max}=1.5$ pour $\theta=\frac{\pi}{2}$ (Hertzian dipole).

  In dB we have $D_{\max}(\text{dB})=10\log(D_{\max})=1.76$ dB.

- for $n>0$

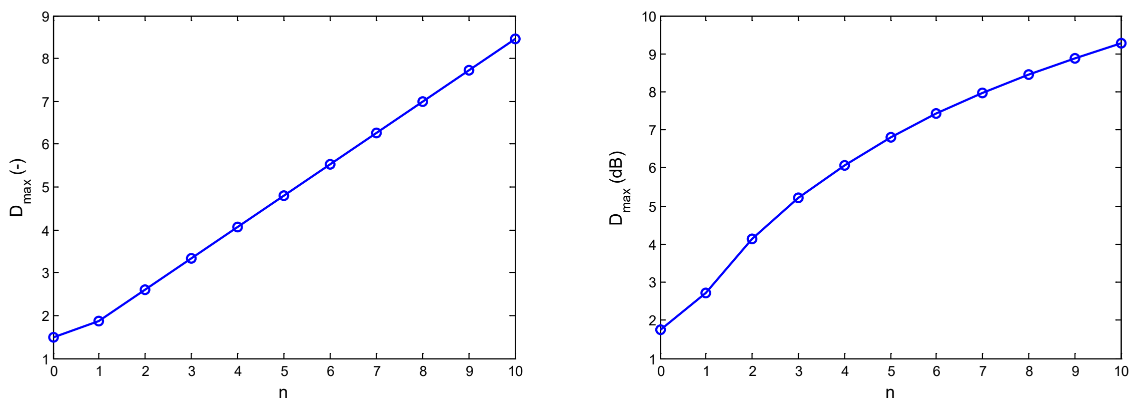
  We look for teh maxima of the function $D(\theta)$:

$$\frac{dD(\theta)}{d\theta} = \frac{2\sin\theta\cos^{2n-1}\theta\left(\cos^2\theta - n\sin^2\theta\right)}{\frac{1}{2n+1} - \frac{1}{2n+3}}$$

We see that the zeros of $\sin\theta$ et de $\cos^{2n-1}\theta$ are minima, while the maxima are given by the zeros of $\left[\cos^2\theta - n\sin^2\theta\right]$. The angles corresponding to the maxima are then given by $\tan^2\theta = \frac{1}{n} \Rightarrow \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{n}{n+1}$.

Introducing this in the expression for D, we obtain $D_{\max}$ (as a function of n):

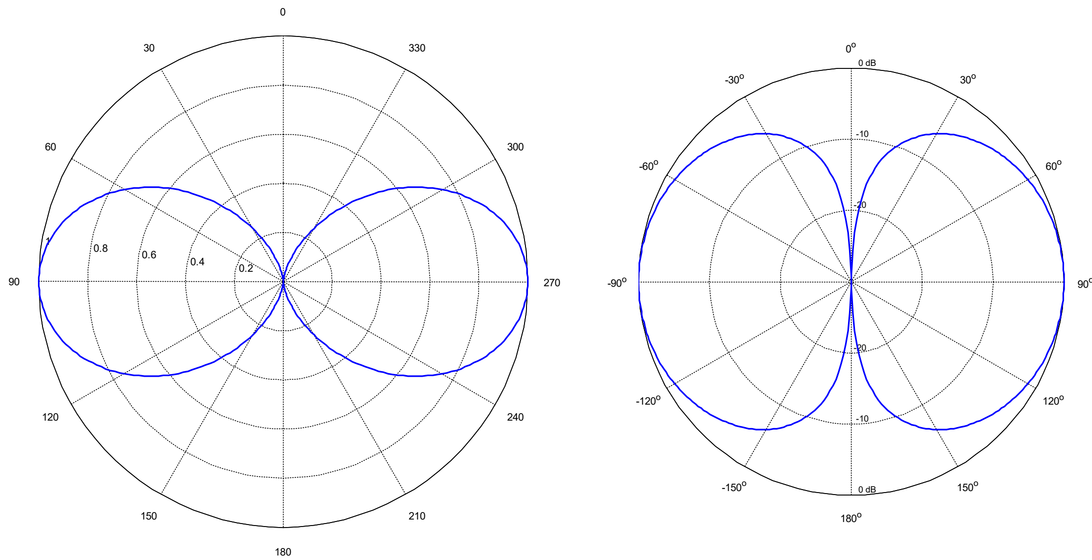
$$D_{\max} = \frac{1}{\frac{1}{2n+1} - \frac{1}{2n+3}} \cdot \frac{n^n}{(n+1)^{n+1}}$$



(a) (b)

Figure 1: $D_{\max}$ as a function of n: (a) linear scale, (b) scale in dB: $D_{\max}(\text{dB}) = 10\log(D_{\max})$.

The power radiation patterns are given in figure 2 for the sake of information


0
30
330
60
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90
270
120
240
150
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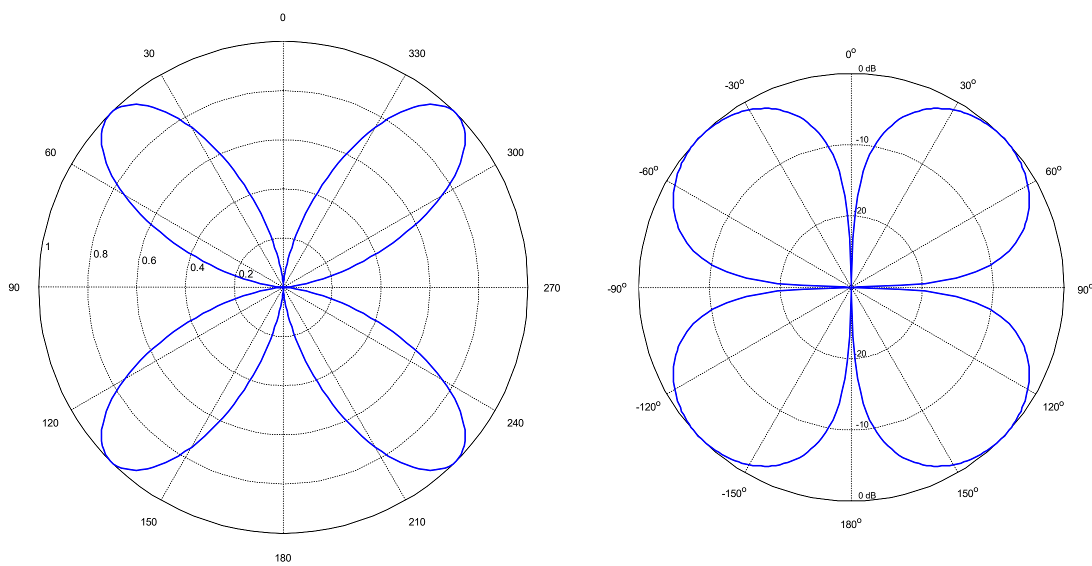
180
1
0.8
0.6
0.4
0.2
0°
0 dB
-30°
30°
-60°
60°
-10
-20
-90°
90°
-120°
120°
-150°
150°
180°

$n = 0$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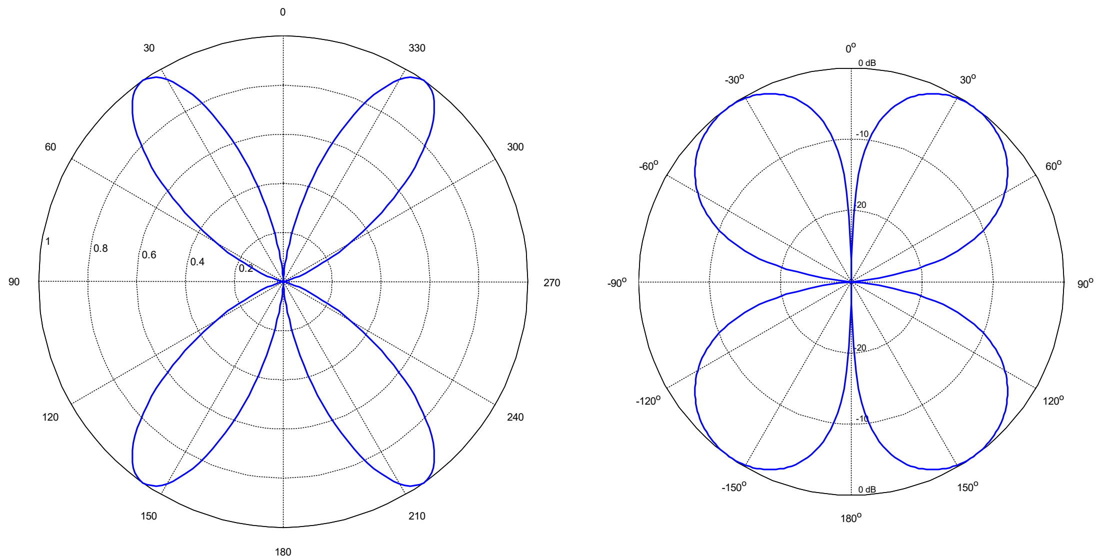
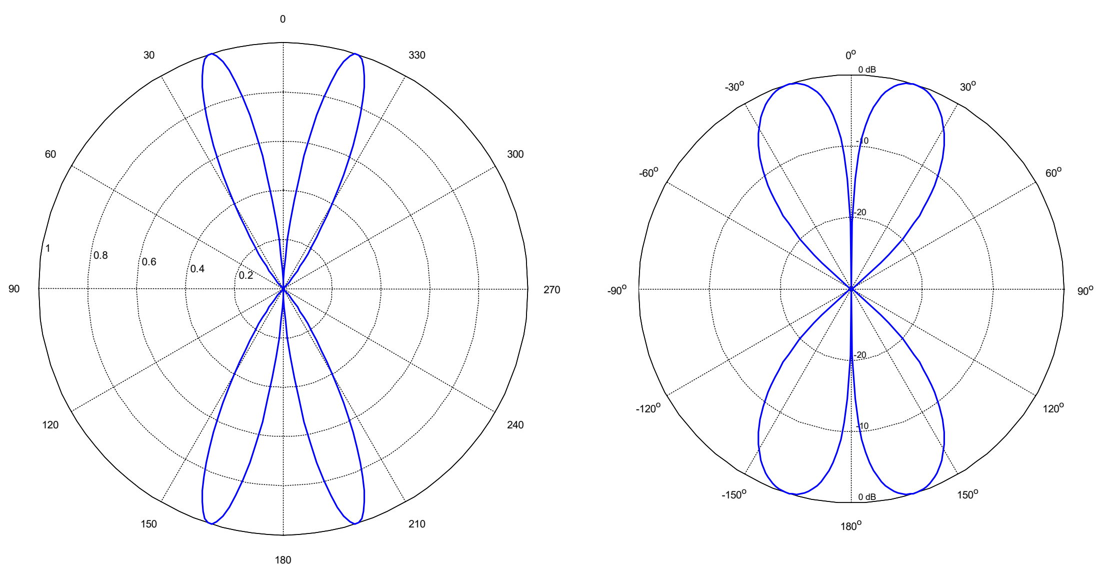
0°
0 dB
-30°
30°
-60°
60°
-10
-20
-90°
90°
-120°
120°
-150°
150°
180°

$n = 1$



n = 2



n = 10

(a) (b)

Figure 2: power radiation pattern a) linear scale, b) in dB